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Orthogonal Polynomials on the Unit Circle: Symmetrization and Quadratic Decomposition

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Problems related to the symmetrization of sequences of orthogonal polynomials on the real line play an important role, as shown in Chihara ("An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978), pages 40–43. On the other hand, the study done in Chihara and Chihara (*J. Math. Anal. Appl.* **126** (1987), 275–291) corresponds to a particular quadratic decomposition of a sequence of orthogonal polynomials on the real line, as a constructive method of a class of nonsymmetric orthogonal polynomials. In this paper we present some results concerning the symmetrization and quadratic decomposition of sequences of orthogonal polynomials, related to a quasi-definite functional on the unit circle.

1. INTRODUCTION

Let $\mathscr{T}$ be an infinite Hermitian Toeplitz matrix; that is, $\mathscr{T} = (c_{i-j})_{i,j=0}^{\infty}$ with $c_{i-j} = \overline{c_{j-i}}$. If we denote

$$\mathscr{T}_n = (c_{i-j})_{i,j=0}^{n}$$

and assume $\Delta_n = \det \mathscr{T}_n \neq 0$ for all n, it is well known (see [3]) that the polynomials (φ_n) defined by

$$\varphi_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_{-1} \\ 1 & z & \cdots & z^n \end{vmatrix} \qquad n \geqslant 1 \tag{1.1}$$

$$\varphi_0(z) = 1$$

640/65/1-8

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are orthogonal with respect to the linear functional σ, on the linear space of polynomials, defined by $\sigma(z^n) = c_n$ and extended by linearity to all polynomials.

Note that in case $\Delta_n > 0$ for all n, there exists a positive measure μ on the unit circle (T) such that the moments c_k have the integral representation

$$c_k = \int_T e^{ik\theta} \, d\mu(\theta).$$

Although the general case of the moment problem is unsolved, the sequence (φ_n) given by (1.1) is usually called orthogonal on the unit circle.

If we define the *-operator so that

$$\varphi_n^*(z) = z^n \overline{\varphi_n}\left(\frac{1}{z}\right)$$

then the polynomials in (1.1) are connected by

$$\begin{aligned} &\varphi_n(z) = z\varphi_{n-1}(z) + \varphi_n(0)\, \varphi_{n-1}^*(z) \qquad n \geqslant 1 \\ &\varphi_0(z) = 1 \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &\varphi_n^*(z) = (1 - |\varphi_n(0)|^2)\, \varphi_{n-1}^*(z) + \overline{\varphi_n(0)}\, \varphi_n(z) \qquad n \geqslant 1 \\ &\varphi_0(z) = 1, \end{aligned} \tag{1.3}$$

which are the relationships verified for orthogonal polynomials on the unit circle when the measure μ exists.

In this work we are going to study the symmetrization problem and the quadratic decomposition for a sequence of monic orthogonal polynomials (MOPS) defined on the unit circle (T) in the sense we have explained above: we consider a Toeplitz matrix with nonzero principal minors or equivalently a sequence of monic polynomials verifying (1.2) or (1.3) with the additional condition of regularity which can be expressed by $|\varphi_n(0)| \neq 1$ for all n. This is a consequence of

$$1 - |\varphi_n(0)|^2 = \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2} \neq 0. \tag{1.4}$$

The parameters $\varphi_n(0)$ are called Szegö–Schur of reflection parameters and they play an important role, because they characterize completely and uniquely the MOPS (φ_n).

The above problems have already been studied on the real line (see [1, 2, 4]).

The main results we are giving in this paper are:

1. Let (φ_n) be a MOPS on T. Then

(a) There exists one and only one MOPS(ϕ_n) on T such that $\phi_{2n}(z) = \varphi_n(z^2)$. Furthermore, $\phi_{2n+1}(z) = z\varphi_n(z^2)$.

(b) There exists one and only one MOPS(ϕ_n) on T such that $\phi_{2n+1}(z) = z\varphi_n(z^2)$. Furthermore, $\phi_{2n}(z) = \varphi_n(z^2)$.

2. Let us consider the quadratic decomposition of (φ_n):

$$\varphi_{2n}(z) = A_n(z^2) + zB_{n-1}(z^2) \qquad n \geqslant 1$$

$$\varphi_{2n+1}(z) = C_n(z^2) + zD_n(z^2) \qquad n \geqslant 0.$$

(a) Given (A_n) a MOPS on T, all the MOPS (φ_n) on T with (A_n) as even component are obtained.

(b) In a similar way, all the MOPS (φ_n) on T with known odd component (D_n) are obtained.

(c) If (A_n) is a MOPS in T then for all n, except at most one, $\varphi_{2n+1}(0) = 0$. In this case $D_n = A_n$ for all natural integers n.

(d) If either $\varphi_{2n+1}(0) = 0$ for all n or if there exists n such that $\varphi_{2n+1}(0) \neq 0$ and $\varphi_{2n}(-z) = \varphi_{2n}(z)$ then (A_n) is a MOPS on T. Furthermore, $D_n = A_n$ for all natural integers n.

2. Symmetrization

THEOREM 2.1. *Let (φ_n) be a MOPS on* T. *Then there exists one and only one MOPS (ϕ_n) on* T *such that*

$$\phi_{2n}(z) = \varphi_n(z^2) \qquad n \geqslant 0.$$

Furthermore

$$\phi_{2n+1}(z) = z\varphi_n(z^2) \qquad n \geqslant 0.$$

Proof. Given (φ_n) we consider the sequence (ϕ_n) such that $\phi_{2n}(z) = \varphi_n(z^2)$. We will show (ϕ_n) is a MOPS under the conditions of the theorem. The uniqueness will be a consequence of the uniqueness of the reflection parameters and so wil be the existence. Let us write down the recurrence relation for the φ_n's

$$\varphi_n(z) = (1 - |\varphi_n(0)|^2)\, z\varphi_{n-1}(z) + \varphi_n(0)\, \varphi_n^*(z) \qquad n \geqslant 1. \qquad (2.1)$$

Note that it is the reciprocal of (1.3) through the *-operator. After replacement of z by z^2 this yields

$$\phi_{2n}(z) = (1 - |\phi_{2n}(0)|^2)\, z^2 \phi_{2n-2}(z) + \phi_{2n}(0)\, \phi_{2n}^*(z) \qquad n \geqslant 1. \tag{2.2}$$

On the other hand, if (ϕ_n) is a MOPS,

$$\phi_{2n}(z) = (1 - |\phi_{2n}(0)|^2)\, z\phi_{2n-1}(z) + \phi_{2n}(0)\, \phi_{2n}^*(z) \qquad n \geqslant 1. \tag{2.3}$$

Comparing in (2.2) and (2.3) the expressions for $\phi_{2n}(z) - \phi_{2n}(0)\, \phi_{2n}^*(z)$, the desired result

$$\phi_{2n-1}(z) = z\varphi_{n-1}(z^2)$$

is obtained. As we said at the beginning of the proof, by consideration of the sequence

$$1\ ,\ 0\ ,\ \varphi_1(0)\ ,\ 0\ ,\ \varphi_2(0)\ ,\ \ldots, \tag{2.4}$$

there is a unique MOPS on T such that its reflection parameters are the ones given in (2.4). ▮

Remarks. 1. Given a MOPS on T, through the above symmetrization, a new MOPS can be generated.

2. The odd reflection parameters for the new MOPS, $\phi_n(0)$, are zero.

THEOREM 2.2. *Let (φ_n) be a MOPS on* T. *Then there exists one and only one MOPS (ϕ_n) on* T *such that*

$$\phi_{2n+1}(z) = z\varphi_n(z^2) \qquad n \geqslant 0.$$

Furthermore

$$\phi_{2n}(z) = \varphi_n(z^2) \qquad n \geqslant 0.$$

Proof. Let (ϕ_n) be a MOPS on T such that $\phi_{2n+1}(z) = z\varphi_n(z^2)$. It follows that for all $n \geqslant 0$, $\phi_{2n+1}(0) = 0$, and from the recurrence relation (1.2) written for ϕ_{2n+1},

$$\phi_{2n+1}(z) = z\phi_{2n}(z),$$

and from the hypothesis we deduce that

$$\phi_{2n}(z) = \varphi_n(z^2).$$

The reflection parameters will be

$$1\ ,\ 0\ ,\ \varphi_1(0)\ ,\ 0\ ,\ \varphi_2(0)\ ,\ \ldots.$$

It is known that a MOPS on T is uniquely determined by its reflection parameters. It is clear that the sequence

$$\phi_{2n+1}(z) = z\varphi_n(z^2) \qquad n \geqslant 0$$

$$\phi_{2n}(z) = \varphi_n(z^2) \qquad n \geqslant 0$$

satisfies a recurrence relation and its reflection parameters are the above mentioned. ∎

Remarks. It is interesting to recall that in the real case if (P_n) is a MOPS and we look for the (Q_n) which are MOPS and such that $Q_{2n+1}(x) = xP_n(x^2)$, there is not a unique solution. Nevertheless the difference between the linear functionals corresponding to two solutions is $\lambda\delta(x)$, where λ is a complex number.

3. Quadratic Decomposition

In this section we consider the quadratic decomposition of (φ_n), a MOPS on T. We are interested in recurrence properties for the sequences involved in the decomposition.

Let us write down the even and odd terms of φ_n as follows:

$$\varphi_{2n}(z) = A_n(z^2) + zB_{n-1}(z^2) \qquad n \geqslant 1 \tag{3.1}$$

$$\varphi_{2n+1}(z) = C_n(z^2) + zD_n(z^2) \qquad n \geqslant 0. \tag{3.2}$$

A_n and D_n are monic polynomials of degree n and the polynomials B_n and C_n are of degree less than or equal to n. We are going to show that the sequences (B_n), (C_n), and (D_n) can be expressed in terms of (A_n).

Lemma 3.1. *The sequences (A_n), (B_n), (C_n), and (D_n) in the quadratic decomposition* (3.1) *and* (3.2) *are related by the formulas*

$$zD_{n-1}(z) = \frac{A_n(z) - A_n(0)\,A_n^*(z)}{1 - |A_n(0)|^2}$$

$$\overline{C_n(0)}\,zB_{n-1}(z) = \frac{A_{n+1}^*(z) - (1 - |A_{n+1}(0)|^2)\,A_n^*(z) - \overline{A_{n+1}(0)}\,A_{n+1}(z)}{1 - |A_{n+1}(0)|^2}$$

$$\overline{C_n(0)}\,C_n(z) = (|C_n(0)|^2 - 1)\,A_n^*(z) + \frac{A_{n+1}^*(z) - \overline{A_{n+1}(0)}\,A_{n+1}(z)}{1 - |A_{n+1}(0)|^2}.$$

Proof. By using the recurrence relation (1.2) for φ_{2n} and identities (3.1) and (3.2), we obtain, after identifying odd and even components,

$$A_n(z) = zD_{n-1}(z) + A_n(0)\, D^*_{n-1}(z) \tag{3.3}$$

$$B_{n-1}(z) = C_{n-1}(z) + A_n(0)\, C^*_{n-1}(z). \tag{3.4}$$

In a similar way, by using the recurrence relation for φ_{2n-1} and again identifying odd and even components, we obtain

$$C_n(z) = zB_{n-1}(z) + C_n(0)\, A^*_n(z) \tag{3.5}$$

$$D_n(z) = A_n(z) + C_n(0)\, B^*_{n-1}(z). \tag{3.6}$$

The four above written equations are valid for $n \geqslant 1$ and with initial conditions

$$D_0(z) = A_0(z) = 1, \quad C_0(z) = \varphi_1(0), \quad B_0(z) = \varphi_1(0) + \overline{\varphi_1(0)}\, \varphi_2(0).$$

The reciprocal equation of (3.3) (obtained by applying the *-operator) is

$$A^*_n(z) = D^*_{n-1}(z) + \overline{A_n(0)}\, zD_{n-1}(z). \tag{3.7}$$

Hence

$$zD_{n-1}(z) = \frac{A_n(z) - A_n(0)\, A^*_n(z)}{1 - |A_n(0)|^2} \tag{3.8}$$

Take note that $A_n(0) = \varphi_{2n}(0)$ and for the sequence (φ_n) the reflection parameters are, in modulus, different from 1.

Introducing (3.8) in (3.6) and applying, once again, the *-operator, we obtain

$$\overline{C_n(0)}\, zB_{n-1}(z) = \frac{A^*_{n+1}(z) - (1 - |A_{n+1}(0)|^2)\, A^*_n(z) - \overline{A_{n+1}(0)}\, A_{n+1}(z)}{1 - |A_{n+1}(0)|^2}. \tag{3.9}$$

If $C_n(0) \neq 0$ (that is, if $\varphi_{2n+1}(0) \neq 0$), Eq. (3.9) gives a representation of B_n in terms of the sequence (A_n).

For those $m \in \mathbf{N}$ such that $C_m(0) = 0$ we have from (3.5) and (3.6)

$$C_m(z) = zB_{m-1}(z)$$

$$D_m(z) = A_m(z);$$

from (3.3) and (3.4)

$$D_m(z) = zD_{m-1}(z) + D_m(0)\, D^*_{m-1}(z)$$

$$B_m(z) = zB_{m-1}(z) + A_{m+1}(0)\, B^*_{m-1}(z).$$

We obtain B_m in terms of B_{m-1}.

If $C_{m-1}(0) = 0$, then B_{m-1} can be written in terms of B_{m-2} and so on.

Finally, if $C_n(0) \neq 0$, multiplying in (3.5) by $\overline{C_n(0)}$ and taking into account Eq. (3.9) we get

$$\overline{C_n(0)}\, C_n(z) = (|C_n(0)|^2 - 1)\, A_n^*(z) + \frac{A_{n+1}^*(z) - \overline{A_{n+1}(0)}\, A_{n+1}(z)}{1 - |A_{n+1}(0)|^2}, \tag{3.10}$$

which gives (C_n) in terms of (A_n). ∎

By using the above Lemma we will characterize all the sequences (φ_n) for whom the even component (A_n) is a MOPS on T.

We consider the recurrence relation

$$A_n(z) - A_n(0)\, A_n^*(z) = (1 - |A_n(0)|^2)\, zA_{n-1}(z) \qquad n \geqslant 1 \tag{3.11}$$

obtained from (1.2) and the *-operator. Comparing with Eq. (3.8) we conclude that

$$\forall n \geqslant 0: \qquad A_n = D_n. \tag{3.12}$$

We observe that the right hand side of Eq. (3.9) is zero because it is the recurrence relation (1.3) for the sequence (A_n). Thus, we have

$$\overline{C_n(0)}\, zB_{n-1}(z) = 0 \qquad n \geqslant 1. \tag{3.13}$$

Finally, comparing with Eq. (3.10),

$$\overline{C_n(0)}\, C_n(z) = |C_n(0)|^2\, A_n^*(z) \qquad n \geqslant 1. \tag{3.14}$$

We analyze now two situations:

(1) There exists a $k \in \mathbf{N}$ such that $C_k(0) \neq 0$.

(2) For all $n \in \mathbf{N}$ $C_n(0) = 0$.

(1) Let k be the smallest index such that $C_k(0) \neq 0$ then, from (3.13), $B_{k-1}(z) = 0$ and from (3.5), $C_k(z) = C_k(0)\, A_k^*(z)$.

Equation (3.4) gives

$$C_{k-1}(z) + A_k(0)\, C_{k-1}^*(z) = 0.$$

Therefore $C_{k-1}(z) = 0$. Hence, by evaluating (3.5) in $n = k-1$ it follows that $B_{k-2}(z) = 0$. By doing so we arrive at

$$\forall m < k \qquad C_m(z) = B_m(z) = 0$$

Together with (3.12) and substituting in (3.1) and (3.2) we obtain

$$\begin{aligned} \forall m < k \qquad \varphi_{2m}(z) &= A_m(z^2) \\ \varphi_{2m+1}(z) &= zA_m(z^2). \end{aligned} \tag{3.15}$$

It is clear that there is only one k such that $C_k(0) \neq 0$, if it exists at all. If j is an index such that $C_j(0) \neq 0$, the same argument leads to $C_m(z) = 0$ provided m is less than j, but $k < j$ and then $C_k(0) = 0$ against our hypothesis.

The obtained relations for k are

$$\begin{aligned} \varphi_{2k}(z) &= A_k(z^2) \\ \varphi_{2k+1}(z) &= zA_k(z^2) + C_k(0)\, A_k^*(z^2) \end{aligned} \tag{3.16}$$

and for $n > k$

$$\begin{aligned} \varphi_{2n}(z) &= A_n(z^2) + zB_{n-1}(z^2) \\ \varphi_{2n+1}(z) &= zA_n(z^2) + z^2 B_{n-1}(z^2). \end{aligned} \tag{3.17}$$

(2) Let us assume for all n, $C_n(0) = 0$. Then

$$C_n(z) = zB_{n-1}(z) \qquad \text{and} \qquad D_n(z) = A_n(z) \tag{3.18}$$

Substituting in (3.1) and (3.2)

$$\begin{aligned} \varphi_{2n}(z) &= A_n(z^2) + zB_{n-1}(z^2) \\ \varphi_{2n+1}(z) &= z^2 B_{n-1}(z^2) + zA_n(z^2) \end{aligned} \tag{3.19}$$

and substituting in (3.4)

$$B_n(z) = zB_{n-1}(z) + A_{n+1}(0)\, B_{n-1}^*(z). \tag{3.20}$$

It is clear from Eqs. (3.18), (3.19), and (3.20) that we can construct (B_n), (C_n), and (D_n) from (A_n).

Furthermore, if $B_0(z) \neq 0$ then $B_n(z)/B_0(z)$ is a MOPS with $B_n(0) = A_{n+1}(0)$. This corresponds to a backward shift in the reflection parameters sequence.

If $B_0(z) = 0$ from (3.20) it is clear that $B_n(z) = 0$ for all n. We can express (3.19)

$$\begin{aligned} \varphi_{2n}(z) &= A_n(z^2) \\ \varphi_{2n+1}(z) &= zA_n(z^2). \end{aligned} \tag{3.21}$$

Let us now assume (D_n) is a MOPS on T. From Eq. (3.3) and recurrence relation held by the D_n's,

$$A_n(z) - D_n(z) = [A_n(0) - D_n(0)]\, D_{n-1}^*(z).$$

Comparing with Eq. (3.6)

$$C_n(0)\, B_{n-1}^*(z) = [D_n(0) - A_n(0)]\, D_{n-1}^*(z)$$

and applying the *-operator

$$\overline{C_n(0)}\, B_{n-1}(z) = [\overline{D_n(0)} - \overline{A_n(0)}]\, D_{n-1}(z). \tag{3.22}$$

There are two different cases:

(1) $D_n(0) = A_n(0)$ for all n.

(2) There exists n such that $D_n(0) \neq A_n(0)$.

(1) $D_n(0) = A_n(0)$ together with (3.3) gives $D_n(z) = A_n(z)$ for all n. Since $\overline{C_n(0)}\, B_{n-1}(z) = 0$ then $C_n(0) = 0$ for all n or there exists a k such that $C_k(0) \neq 0$.

If $C_n(0) = 0$ for all n, then from (3.5) $C_n(z) = zB_{n-1}(z)$, $n \geqslant 1$. Substituting in (3.4)

$$B_n(z) = zB_{n-1}(z) + A_{n-1}(0)\, B^*_{n-1}(z)$$

we obtain, as expected, the same results we got before; see Eq. (3.20).

If $C_k(0) \neq 0$ then $B_{k-1}(z) = 0$ and from (3.4) we have

$$C_{k-1}(z) + A_k(0)\, C^*_{k-1}(z) = 0,$$

which gives $C_{k-1}(z) = 0$. Now, from (3.5) it follows that $B_{k-2}(z) = 0$ and inductively we conclude that $C_0(z) = B_0(z) = 0$.

It follows trivially that k, if it exists, is unique.

(2) $D_n(0) \neq A_n(0)$. Therefore $C_n(0) \neq 0$ and $\deg B_{n-1} = n-1$.

From Eq. (3.4) and its reciprocal we obtain

$$C_{n-1}(z) = \frac{B_{n-1}(z) - A_n(0)\, B^*_{n-1}(z)}{1 - |A_n(0)|^2}. \tag{3.23}$$

(3.22) and (3.23) together give

$$C_{n-1}(z) = \frac{\dfrac{\overline{D_n(0)} - \overline{A_n(0)}}{\overline{C_n(0)}}\, D_{n-1}(z) - A_n(0)\, \dfrac{D_n(0) - A_n(0)}{C_n(0)}\, D^*_{n-1}(z)}{1 - |A_n(0)|^2}.$$

As a consequence of the above results, the following theorem holds

THEOREM 3.2. *Let* (φ_n) *be a MOPS on* T *and consider the quadratic decomposition given in* (3.1) *and* (3.2). *If* (A_n) *is a MOPS on* T, *then for all n, except at most one,* $\varphi_{2n+1}(0) = 0$. *Moreover,* $D_n = A_n$ *for all natural integer n.*

Proof. Note that $\varphi_{2n+1}(0) = C_n(0)$. We have just shown that if (A_n) is a MOPS then $C_n(0) \neq 0$ at most for one value of n and $A_n = D_n$. ∎

As a partial converse of this theorem, we have

THEOREM 3.3. *With the notation given above, either if $\varphi_{2n+1}(0) = 0$ for all n or if there exists exactly one n such that $\varphi_{2n+1}(0) \neq 0$ and $\varphi_{2n}(-z) = \varphi_{2n}(z)$ then (A_n) is a MOPS on T. Furthermore, $D_n = A_n$ for all n.*

Proof. We analyze the two possible cases:

(a) $\varphi_{2n+1}(0) = 0$ for all natural integer n.

(b) There exists m such that $\varphi_{2m+1}(0) \neq 0$.

(a) In this case $C_n(0) = 0$ and from Eqs. (3.5) and (3.6) we obtain

$$D_n(z) = A_n(z) \quad \text{and} \quad C_n(z) = zB_{n-1}(z).$$

From Eq. (3.3) with initial condition $A_0(z) = 1$ it follows that (A_n) is a MOPS on T.

(b) $C_n(0) = 0$ for all $n \neq m$ and $C_m(0) \neq 0$.

The two following relations come from (3.3):

$$\forall n \neq m+1 \qquad A_n(z) = zA_{n-1}(z) + A_n(0)\, A^*_{n-1}(z) \tag{3.24}$$

$$A_{m+1}(z) = zD_m(z) + A_{m+1}(0)\, D^*_m(z). \tag{3.25}$$

Substituting the value of $D_n(z)$ from (3.6) into (3.25),

$$\begin{aligned} A_{m+1}(z) &= z\{A_m(z) + C_m(0)\, B^*_{m-1}(z)\} \\ &\quad + A_{m+1}(0)\{A^*_m(z) + \overline{C_m(0)}\, zB_{m-1}(z)\} \\ &= zA_m(z) + A_{m+1}(0)\, A^*_m(z) \\ &\quad + z\{C_m(0)\, B^*_{m-1}(z) + A_{m+1}(0)\, \overline{C_m(0)}\, B_{m-1}(z)\}. \end{aligned} \tag{3.26}$$

But

$$z\{C_m(0)\, B^*_{m-1}(z) + A_{m+1}(0)\, \overline{C_m(0)}\, B_{m-1}(z)\} \tag{3.27}$$

vanishes, Eqs. (3.24) and (3.26) show (A_n) is a MOPS on T, because of $|A_n(0)| \neq 1$. ∎

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